ANALYTICAL SYNTHESIS OF CONTROLLER TRANSFER MATRICES ON THE BASIS OF FREQUENCY DOMAIN PERFORMANCE CRITERIA. I

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Concepts of frequency domain performance indices of systems with m controllers are introduced. A synthesis problem of controller transfer matrices is formulated. A theorem underlying the solution of this problem is proved.

INTRODUCTION

Systems with several controllers and a number of controlled quantities are widely used in control engineering. In particular, stabilization systems of aircraft and gyro-platforms, especially in cases where the interaction of stabilization channels cannot be neglected, belong to such systems. Basic results of the frequency-domain synthesis theory [1, 2] apply to the case of a single controller.

In this paper we present a formulation and solution of the synthesis problem of the controller transfer matrix on the basis of frequency domain performance indices.

Papers on the theory of analytical design of optimal controllers [3] and those on the study of frequency domain properties of optimal systems [4, 5] were the starting point of the investigation whose results are presented here.

1. FORMULATION OF THE BASIC PROBLEM

We consider the control system described by the equations

$$\left\|\begin{matrix} Q_{11}(s)\,Q_{12}(s)\ldots Q_{1\rho}(s) \\ Q_{21}(s)\,Q_{22}(s)\ldots Q_{2\rho}(s) \\ \vdots \\ Q_{\rho1}(s)\,Q_{\rho2}(s)\ldots Q_{\rho\rho}(s) \end{matrix}\right\| \times \left\|\begin{matrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{matrix}\right\| = \left\|\begin{matrix} N_{11}(s)\,N_{12}(s)\ldots N_{1m}(s) \\ N_{21}(s)\,N_{22}(s)\ldots N_{2m}(s) \\ \vdots \\ N_{\rho1}(s)\,N_{\rho2}(s)\ldots N_{\rho m}(s) \end{matrix}\right\| \times \left\|\begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix}\right\| + \left\|\begin{matrix} L_{11}L_{12}\ldots L_{1\mu} \\ L_{21}L_{22}\ldots L_{2\mu} \\ \vdots \\ L_{\rho1}L_{\rho2}\ldots L_{\rho\mu} \end{matrix}\right\| \times \left\|\begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_\mu \end{matrix}\right\|, \qquad (1.1)$$

$$\left\|\begin{matrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{matrix}\right\| = \left\|\begin{matrix} \bar{d}_{11}\bar{d}_{12}\ldots \bar{d}_{1\rho} \\ \bar{d}_{21}\bar{d}_{22}\ldots \bar{d}_{2\rho} \\ \vdots \\ \bar{d}_{r1}\bar{d}_{r2}\ldots,\bar{d}_{r\rho} \end{matrix}\right\| \times \left\|\begin{matrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{matrix}\right\|, \qquad (1.2)$$

$$\left\|\begin{matrix} y_1^* \\ y_2^* \\ \vdots \\ y_\delta^* \end{matrix}\right\| = \left\|\begin{matrix} \bar{d}_{11}^*\bar{d}_{12}^*\ldots \bar{d}_{1\rho}^* \\ \bar{d}_{21}^*\bar{d}_{22}^*\ldots \bar{d}_{2\rho}^* \\ \vdots \\ \bar{d}_{\delta1}^*\bar{d}_{\delta2}^*\ldots \bar{d}_{\delta\rho}^* \end{matrix}\right\| \times \left\|\begin{matrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{matrix}\right\|, \qquad (1.3)$$

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$$\begin{Vmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Vmatrix} = \begin{Vmatrix} \frac{r_{11}(s)}{d_1(s)} & \frac{r_{12}(s)}{d_1(s)} & \cdots & \frac{r_{1r}(s)}{d_1(s)} \\ \frac{r_{21}(s)}{d_2(s)} & \frac{r_{22}(s)}{d_2(s)} & \cdots & \frac{r_{2r}(s)}{d_2(s)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{r_{m1}(s)}{d_m(s)} & \cdots & \cdots & \frac{r_{mr}(s)}{d_m(s)} \end{Vmatrix} \times \begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{Vmatrix}. \quad (1.4)$$

Equations (1.1)-(1.3) describe the plant. Here $z_i(t)$ ($i = 1, \ldots, \rho$) are the coordinates of the plant, $y_i(t)$ ($i = 1, \ldots, r$) are the variables of the plant accessible to direct measurement, $y_i^*(t)$ ($i = 1, \ldots, \delta$) are the controlled variables ("outputs" of the plant), $u_i(t)$ ($i = 1, \ldots, m$) are the coordinates of the controllers ("inputs" of the plant); $f_i(t)$ ($i = 1, \ldots, \mu$) are external disturbances, and $s = d/dt$ (in addition, the letter s is used also as a symbol of Laplace transformation for zero initial conditions);

$$Q_{\nu\beta}(s) = \sum_{i=0}^{a_{\nu\beta}} q_{\nu\beta}^i s^i \quad (\nu, \beta = 1, \ldots, \rho), \qquad N_{\nu\beta}(s) = \sum_{i=0}^{b_{\nu\beta}} n_{\nu\beta}^i s^i \quad (\nu = 1, \ldots, \rho;\ \beta = 1, \ldots, m),$$

where $a_{\nu\beta}$, $b_{\nu\beta}$ are the degrees of the polynomials $Q_{\nu\beta}(s)$ ($\nu, \beta = 1, \ldots, \rho$) and $N_{\nu\beta}(s)$ ($\nu = 1, \ldots, \rho$; $\beta = 1, \ldots, m$), respectively.

Equations (1.4) are the controller equations, in which

$$r_{\nu\beta}(s) = \sum_{i=0}^{l_{\nu\beta}} r_{\nu\beta}^i s^i \quad (\nu = 1, \ldots, m;\ \beta = 1, \ldots, r), \; d_\nu(s) = \sum_{i=0}^{t_\nu} d_\nu^i s^i \; (\nu = 1, \ldots, m),$$

where $l_{\nu\beta}$, t_ν are the degrees of the polynomials $r_{\nu\beta}(s)$ and $d_\nu(s)$ ($\nu = 1, \ldots, m$; $\beta = 1, \ldots, r$), respectively.

The system (1.1)-(1.4) can also be written in the more compact form

$$Q(s)z = N(s)u + Lf, \quad (1.1')$$

$$y = \bar{D}z, \quad (1.2')$$

$$y^* = \bar{D}^*z, \quad (1.3')$$

$$u = R(s)y, \quad (1.4')$$

where $Q(s)$, $N(s)$ are polynomial matrices of dimensions $\rho \times \rho$ and $\rho \times m$ respectively, $R(s)$ is a transfer matrix of dimension $m \times r$, $\bar{D}$, $\bar{D}^*$ are matrices of numbers with the dimensions $r \times \rho$, $\delta \times \rho$.

The transfer matrix $R(s)$ of the controllers is said to be relizable, if among its elements

$$R_{ij}(s) = \frac{r_{ij}(s)}{d_i(s)} \quad (i = 1, \ldots, m;\ j = 1, \ldots, r)$$

there are none with the degree of the numerator greater than the degree of the deonminator. In a contrary case the transfer function $R(s)$ is said to be nonrealizable. According to this definition $R(s)$ is realizable, if the inequality

$$l_{\nu\beta} \leqslant t_\nu \quad (\nu = 1, \ldots, m;\ \beta = 1, \ldots, r) \quad (1.5)$$

is satisfied.

Often only the maximum value $\bar{f}_j$ is known about each component f_i ($i = 1, \ldots, \mu$) of the disturbance vector. We shall study the system (1.1)-(1.4) in the case of typical (or the most unfavorable) action having the form

$$f_i(t) = \begin{cases} \bar{f}_i = \text{const} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leqslant 0. \end{cases} \quad (1.6)$$

We shall use the frequency domain performance indices [1, 2] to assess the performance of the system (1.1)-(1.4).

For this represent the system (1.1)-(1.4) in the form of m equivalent systems. Each of these systems (for example, the ν-th system) consists of a "plant" [which is the plant (1.1) closed by m-1 controllers] having the ν-th controller.

The usual concepts of the transfer function $w_\nu(s)$ of an open-loop system – the phase margin $\varphi_{m\nu}$, the gain margin L_ν, the oscillation index M_ν and the crossover frequency $\omega_{c\nu}$ – are applicable to each of such equivalent systems.

The transfer function $w_\nu(s)$ is called the transfer function of the system (1.1)-(1.4) opened at the ν-th input of the plant, while the values of $\varphi_{m\nu}$, L_ν, M_ν are called the performance indices of the ν-th controller ($\nu = 1, \ldots, m$).

Side by side with these performance indices of multidimensional systems with m controllers we shall use the generalized frequency domain performance indices introduced in [5]. These performance indices [connected with the characteristic equations of the system (1.1)-(1.4) in open-loop (all inputs are open) and closed loop states, $D_{op}(s)$ and $D_{cl}(s)$ respectively] are given by relationships

$$\varphi_m^{ge} = \pi + \arg w_e(j\omega_c^{ge}),\ L = \min\left\{|\operatorname{Re} w_e(j\omega_1)|\frac{1}{|\operatorname{Re} w_e(j\omega_2)|}\right\},$$
$$M = \max_{0<\omega<\infty} \frac{|w_e(j\omega)|}{|1 + w_e(j\omega)|}, \tag{1.7}$$

where $w_e(s)$, ω_1, ω_2, ω_c^{ge} are determined from the equations

$$w_e(j\omega) = \frac{D_{cl}(j\omega)}{D_{op}(j\omega)} - 1,\quad \operatorname{Im} w_e(j\omega_i) = 0\ (i = 1,2),$$
$$\sqrt{w_e(j\omega_c^{ge})\, w_e(-j\omega_c^{ge})} = 1.$$

We call $w_e(s)$ and ω_c^{ge} the generalized function and the generalized crossover frequency of open-loop systems, while the indices (1.7) are called the generalized performance indices. The performance indices (1.7) are poorer in their physical content than the above system of ν-th indices, and in contrast to the latter, they cannot be determined directly from results of an experiment. When m = 1, both systems of frequency domain performance indices introduced here coincide with the usual ones.

The system (1.1)-(1.4) is considered to be "good" with respect to its frequency domain indices, if the conditions

$$\varphi_{mi} \geqslant \varphi_m^*,\ L_i \geqslant L^*,\ M_i \leqslant M^* \quad (i = 1, \ldots, m), \tag{1.8}$$
$$\omega_{ci} \approx \omega_{ci}^{**} \quad (i = 1, \ldots, m) \tag{1.9}$$

or

$$\varphi_c^{ge} \geqslant \varphi_m^*,\ L^{ge} \geqslant L^*,\ M^{ge} \leqslant M^*, \tag{1.10}$$
$$\omega_c^{ge} \approx \omega_c^* \tag{1.11}$$

are fulfilled. Here $\varphi_m^* = 30°$-$45°$, $L^* = 2$-10, $M = 1.5$-2, while ω_c^*, ω_c^{**} are given numbers.

The accuracy indices of a control system usually are the values of the static errors ($y_{i\,st}^*$, $i = 1, \ldots, \delta$) and the dynamic errors ($y_{i\,dyn}^*$, $i = 1, \ldots, \delta$) caused by the action (1.6). In this investigation accuracy of the system (1.1)-(1.4) is established only by values of the static errors, i.e., for this system the inequality

$$|y_{i\,st}^*| \leqslant y_{i\,st}^{**} \quad (i = 1, \ldots, \delta) \tag{1.12}$$

must be satisfied ($y_{i\,st}^{**}$ are given positive numbers).

Problem 1.1. Suppose we have a control system with a plant described by Eqs. (1.1)-(1.3), in which the polynomial matrices Q(s) and N(s) and the matrices of numbers L, $\bar{D}$, $\bar{D}^*$, $\bar{f}$ are given. We have to determine a realizable transfer matrix R(s) of the controllers (1.4) such that the system described by Eqs. (1.1)-(1.4) would be asymptotically stable, that it would satisfy the accuracy requirements (1.12) and the performance requirements (1.8) or (1.10), and that the generalized crossover frequency ω_c^{ge} would be close to a certain ω_c^* given in advance.

The solution of this problem presented below applies to a case where the measured and controlled variables coincide, i.e., $y_i(t) \equiv y_i^*(t)$ $(i = 1, \ldots, r = \delta)$. In addition, the restriction $r = m$ is used to satisfy the requirement (1.12).

2. SOLUTION OF THE BASIC PROBLEM

We shall briefly describe the basic steps of the solution of Problem 1.1.

A formula for reducing Eqs. (1.1) to the Cauchy form is obtained in Appendix 1. After such reduction Eq. (1.1) and the relations (1.2) are written in the form

$$\dot{x} = Px + Bu + Mf, \quad (2.1)$$

$$y = Dx, \quad (2.2)$$

where P, B, M, D are matrices of numbers with the dimensions $n \times n$, $n \times m$, $n \times \mu$, $r \times n$, respectively.

Side by side with the nonhomogeneous system (2.1), (2.2) we shall consider the homogeneous system

$$\dot{x} = Px + Bu, \quad (2.1)$$

$$y = Dx. \quad (2.2)$$

We assume about the properties of the matrices P, B, and D that the equations

$$\text{rank } \|B, PB, \ldots, P^{n-1}B\| = n, \quad (2.3)$$

$$\text{rank } \|D', (DP)', \ldots, (DP^{n-1})'\| = n, \quad (2.4)$$

are satisfied. These equations indicate that the plant (2.1) is completely controllable, and that it is completely observable from the signal (2.2).

We shall present the results of the solution of two problems underlying the solution of Problem 1.1.

Problem 2.1. (The problem of analytical controller design [3].) Suppose there exists a system which is described by the equation

$$\dot{x} = Px + Bu. \quad (2.5)$$

We have to set up a controller equation

$$u = C'x \quad (2.6)$$

such that the system (2.5), (2.6) would be asymptotically stable and that, in addition, the functional

$$I = \int_0^\infty (x'Qx + u'u)\,dt \quad (x'Qx \geqslant 0 \quad \text{for any } x) \quad (2.7)$$

would be minimized for motions of this system with any initial conditions.

The analytical and numerical solutions of this problem are well known [3, 6]. The sought matrix C' (of dimension $m \times n$) is a solution of the system of algebraic equations

$$AP + P'A - ABB'A + Q = 0, \quad (2.8)$$

$$C = -AB \quad (2.9)$$

(A is a positive definite symmetric matrix of dimension $n \times n$).

We note that for the existence of the nontrivial solution $u \equiv [0]$ of Problem 2.1, in the case of nonnegative definite matrices Q, the condition [7]

$$\text{rank } \|H', P'H', \ldots, (P')^{n-1}H'\| = n, \quad (2.10)$$

must be satisfied. Here $Q = H'H$ (H has dimensions $\varkappa \times n$, $\varkappa$ is the rank of the matrix Q).

The result of the solution of the second problem applies to frequency domain properties of optimal systems. We shall formulate it as a theorem.

Theorem 2.1. If we have a control system described by Eqs. (2.5) and (2.6), and its controller has been obtained as a result of solving Problem 2.1 which minimizes the functional (2.7) with an arbitrary nonnegative matrix which satisfies only the condition (2.10), then this system has the following frequency domain properties:

$$\varphi_{m}^{ge} \geqslant 60^\circ,\ L^{ge} \geqslant 2,\ M^{ge} \leqslant 2, \tag{2.11}$$

$$\varphi_{mi} \geqslant 60^\circ,\ L_i \geqslant 2,\ M_i \leqslant 2 \quad (i = 1, \ldots, m). \tag{2.12}$$

A proof of the property (2.11) in the case of a sign-definite matrix Q (i.e., $\varkappa = n$) is presented in [5]. However, the course of this proof is not altered for the case of a nonnegative matrix Q ($\varkappa < n$) which satisfies the condition (2.10).

We shall prove the property (2.12). We write the system (2.5), (2.6) in the form of equivalent systems:

$$\dot{x} = [P + \overset{\nu}{B}\overset{\nu}{C}'] x + B_{[\nu]} u_\nu \ (\nu = 1, \ldots, m), \tag{2.13}$$

$$u_\nu = C'^{[\nu]} x \quad (\nu = 1, \ldots, m), \tag{2.14}$$

where $B_{[\nu]}$ is the ν-th column of the matrix B, $C'^{[\nu]}$ is the ν-th row of the matrix C', and $\overset{\nu}{B}x$ is a matrix of numbers of dimension $n \times (m-1)$, obtained from the matrix B by striking out its ν-th column; $\overset{\nu}{C}'$ is a matrix of numbers of dimension $(m-1) \times n$, obtained from matrix C' by striking out the ν-th row.

The transfer function of each of the systems (2.13), (2.14) has the form

$$w_\nu(s) = -C'^{[\nu]}[Es - P^{\nu}]^{-1} B_{[\nu]} \quad (\nu = 1, \ldots, m), \tag{2.15}$$

where $P^{\nu} = P + \overset{\nu}{B}\overset{\nu}{C}'$ ($\nu = 1, \ldots, m$). Each transfer function $w_\nu(s)$ complies with the definition, introduced earlier, of the transfer function of the system (2.5), (2.6) opened at the ν-th input of the plant.

We consider the ρ-th system of (2.13), (2.14):

$$\dot{x} = [P + \overset{\rho}{B}\overset{\rho}{C}'] x + B_{[\rho]} u_\rho, \tag{2.16}$$

$$u_\rho = C'^{[\rho]} x. \tag{2.17}$$

We note that from optimality of the systems (2.5), (2.6) we have

$$C'^{[\rho]} = \{A B_{[\rho]}\}', \tag{2.18}$$

$$\overset{\rho}{C}' = \{A \overset{\rho}{B}\}', \tag{2.19}$$

where the matrix A having dimension $n \times n$ is the solution of Eq. (2.8).

We now find a controller equation

$$u_\rho = C^{*\prime} x \tag{2.20}$$

such that the functional

$$I = \int_0^\infty [x' (Q + \overset{\rho}{C}\overset{\rho}{C}') x + u_\rho^2]\, dt. \tag{2.21}$$

will be minimized on the solutions of the system (2.16), (2.20).

Following the form (2.8), (2.9), with (2.19) taken into account, we write the system of equations for the sought matrix $C^{*\prime}$:

$$A^* [P - \overset{\rho}{B}\overset{\rho}{B}'A] + [P - \overset{\rho}{B}\overset{\rho}{B}'A]' A^* - A^* B_{[\rho]} B'_{[\rho]} A^* + Q + A\overset{\rho}{B}\overset{\rho}{B}'A = 0, \tag{2.22}$$

$$C^* = -A^* B_{[\rho]}. \tag{2.23}$$

Equation (2.22) can be represented in the form

$$A^* P + P'A^* - A^* \overset{\rho}{B}\overset{\rho}{B}'A - A\overset{\rho}{B}\overset{\rho}{B}'A^* - A^* B_{[\rho]} B'_{[\rho]} A + A\overset{\rho}{B}\overset{\rho}{B}'A + Q = 0. \tag{2.22'}$$

It is obvious that

$$A^* = A \tag{2.24}$$

is one of the solutions of this equation, since for $A^* = A$, Eq. (2.22) coincides with (2.8). Equation (2.22) is satisfi[ed] by the only positive definite matrix A^*. Consequently, this matrix is given by the expression (2.24).

It follows from (2.23), (2.18) that

$$C^* = C^{[\rho]}. \quad (2.2[5])$$

The system (2.16), (2.17) with a single controller (2.17) thus is optimal in the sense of the nonnegative func-tion (2.21). This optimal system [4] possesses the property (2.11) (we recall that for m = 1, $\varphi_m^{ge} = \varphi_m$, $L^{ge} = L$, $M^{ge} = M$).

The above results are valid also for the remaining $m-1$ equivalent systems (2.13), (2.14). Consequently, th[e] properties (2.12) of the optimal system (2.5), (2.6) have been proved.

The subsequent solution of Problem 1.1 rests on the properties (2.11), (2.12) of the optimal system (2.5), (2.6). It consists of setting up the structure and choosing the parameters of the optimization functional such that the con-troller transfer matrix obtained from (2.6) (after substitution of the vector x by a corresponding operator of the vec-tor y) would be realizable and would satisfy the requirements (1.12), (1.11) imposed on the control accuracy and th[e] crossover frequency.

APPENDIX

Reduction of a System of Differential Equations to the Cauchy Form

We write the system (1.1), (1.2) in the form

$$\left(\sum_{i=0}^{\nu} Q^i s^i\right) z = \left(\sum_{i=0}^{\nu-1} N^i s^i\right) u + \left(\sum_{i=0}^{\nu-1} L^i s^i\right) f, \quad (A.1)$$

$$y = \bar{D}z, \quad (A.2)$$

where Q^i ($i = 0, \ldots, \nu$), N^i, L^i ($i = 0, \ldots, \nu$) are matrices of numbers, made up of the coefficients of the correspo[nd]ing powers of s of the matrices Q(s), N(s), L(s).

Equations (A.1) and (A.2) must be reduced to the Cauchy form

$$\dot{x} = Px + Bu + Mf, \quad (A.3)$$

$$y = Dx, \quad (A.4)$$

where P, B, M, D are matrices of numbers having dimensions $n \times n$, $n \times m$, $n \times \mu$, $r \times n$ respectively; x is an n-dimen-sional vector which in the general case has $(\nu\rho - k)$ components. The coefficient k depends on the rank of the ma-trices Q^i ($i = 0, \ldots, \nu$). In particular, for $|Q^\nu| \neq 0$, we have the equation $n = \nu\rho$.

We assume about Eqs. (A.1) that the right side of each of the equations making up the system (A.1) has a pow[er] of s which is at least by one lower than the power of the left side of the same equation. This property of the system (A.1) is called the property "H." We also consider the condition

$$|Q(s)| \not\equiv 0 \quad (A.5)$$

as fulfilled.

The transition from the system (A.1) to the system (A.3) in the case of a nonsingular matrix Q^ν, as well as in the case $N^i \equiv 0$, $L^i \equiv 0$ ($i = 1, \ldots, \nu-1$), is described in [8, 9].

Below transformation into the Cauchy form is considered in a more general case.

Let the rank of the matrix Q^ν be $r_1 < \rho$. Multiplying the separate equations of the system (A.1) by certain co-efficients and adding them to one another, we can represent the coefficient matrix of the system (A.1) in the form

$$Q^\nu = \left\| \begin{matrix} A_1^\nu \\ 0 \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad Q^{\nu-1} = \left\| \begin{matrix} A_1^{\nu-1} \\ A_2^{\nu-1} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \ldots, \quad Q^0 = \left\| \begin{matrix} A_1^0 \\ A_2^0 \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix},$$

$$N^{\nu-1} = \left\| \begin{matrix} B_1^{\nu-1} \\ 0 \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad N^{\nu-2} = \left\| \begin{matrix} B_1^{\nu-2} \\ B_2^{\nu-2} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \ldots, \quad N^0 = \left\| \begin{matrix} B_1^0 \\ B_2^0 \end{matrix} \right\|,$$

$$L^{\nu-1} = \left\| \begin{matrix} C_1^{\nu-1} \\ 0 \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad L^{\nu-2} = \left\| \begin{matrix} C_1^{\nu-2} \\ C_2^{\nu-2} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \ldots, \quad L^0 = \left\| \begin{matrix} C_1^0 \\ C_2^0 \end{matrix} \right\|. \tag{A.6}$$

Here and subsequently we assume that the properties "H" are invariant with respect to equivalent transformations of the system (A.1) which lead to (A.6).

Forming new matrices from (A.6) by interchanging the $(n-r_1)$ last rows of the i-th matrix with the $(n-r_1)$ last rows of the (i + 1)-th matrix $(i = 0, \ldots, \nu)$, we obtain:

$$Q_1^{\nu} = \left\| \begin{matrix} A_1^{\nu} \\ A_2^{\nu-1} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad Q_1^{\nu-1} = \left\| \begin{matrix} A_1^{\nu-1} \\ A_2^{\nu-2} \end{matrix} \right\|, \ldots, \quad Q_0 = \left\| \begin{matrix} A_1^0 \\ 0 \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix},$$

$$N^{\nu-1} = \left\| \begin{matrix} B_1^{\nu-1} \\ B_2^{\nu-2} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad N^{\nu-2} = \left\| \begin{matrix} B_1^{\nu-2} \\ B_2^{\nu-3} \end{matrix} \right\|, \ldots, \quad N^0 = \left\| \begin{matrix} B_1^0 \\ 0 \end{matrix} \right\|, \tag{A.7}$$

$$L^{\nu-1} = \left\| \begin{matrix} C_1^{\nu-1} \\ C_2^{\nu-2} \end{matrix} \right\| \begin{matrix} \}r_1 \\ \}n-r_1 \end{matrix}, \quad L^{\nu-2} = \left\| \begin{matrix} C_1^{\nu-2} \\ C_2^{\nu-3} \end{matrix} \right\|, \ldots, \quad L^0 = \left\| \begin{matrix} C_1^0 \\ 0 \end{matrix} \right\|.$$

We note that the transition from the matrices (A.6) to the matrices (A.7) corresponds to differentiation of the last $n-r_1$ equations of a system of the form (A.1) with matrices (A.6).

The rank of the matrix Q_1^{ν} equals $r_1 + r_2 \geq r_1$. If $r_1 + r_2 < \rho$, then transformations (A.6) and (A.7) must be repeated for the matrices Q_1^i $(i = 0, \ldots, \nu)$, N_1^i, L_1^i $(i = 0, \ldots, \nu - 1)$.

As a result, we obtain the matrices Q_2^i, N_2^i, L_2^i. If the rank of the matrix is less than ρ, then we continue these transformations until the equation

$$\text{rank } Q_j^{\nu} = \rho \tag{A.8}$$

is satisfied.

Similarly to [8], we introduce the new variables

$$\begin{aligned} x^{\nu} &= Q_j^{\nu} z, \\ x^{\nu-1} &= Q_j^{\nu-1} z + Q_j^{\nu} \dot{z} - N_j^{\nu-1} u - L_j^{\nu-1} f, \\ &\vdots \\ x^1 &= Q_j^1 z + Q_j^2 \dot{z} + \ldots + Q_j^{\nu} z^{\nu-1} - N_j^1 u - \ldots - N_j^{\nu-1} u^{\nu-2} - L_j^1 f - \ldots - L_j^{\nu-1} f^{\nu-2}. \end{aligned} \tag{A.9}$$

Eliminating the derivatives z and substituting expressions for x^1 into (A.1), after elimination of z by means of $z = (Q_j^{\nu})^{-1} x^{\nu}$:

$$\begin{aligned} \dot{x}^1 &= -Q_j^0 (Q_j^{\nu})^{-1} x^{\nu} + N_j^0 u + L_j^0 f, \\ \dot{x}^2 &= x^1 - Q_j^1 (Q_j^{\nu})^{-1} x^{\nu} + N_j^1 u + L_j^1 f, \\ &\vdots \\ \dot{x}^{\nu-1} &= x^{\nu-2} - Q_j^{\nu-2} (Q_j^{\nu})^{-1} x^{\nu} + N_j^{\nu-2} u + L_j^{\nu-2} f, \\ \dot{x}^{\nu} &= x^{\nu-1} - Q_j^{\nu-1} (Q_j^{\nu})^{-1} x^{\nu} + N_j^{\nu-1} u + L_j^{\nu-1} f. \end{aligned} \tag{A.10}$$

From the structure of the matrices Q^0, N^0, L^0 in (A.7) we see that the vector x^1 has r_1 independent components, while the derivatives of the remaining $n-r_1$ components are zero. Therefore we neglect these components as linearly independent. Analogously, we conclude that the vector x^2 has $r_1 + r_2$ independent components, and so forth.

The vector x is thus made up of the following independent components:

$$x_1^1, \ldots, x_{r_1}^1, x_1^2, \ldots, x_{r_1+r_2}^2, \ldots, x_1^j, \ldots, x^j_{\sum\limits_{i=1}^{j} r_i}, x_1^{j+1}, \ldots, x_\rho^{j+1}, \ldots, x_1^\nu, \ldots, x_\rho^\nu. \qquad \text{(A.}$$

From (A.11) the system (A.10) can be written as

$$\dot{x} = Px + Bu + Mf, \qquad y = Dx,$$

where

$$P = \begin{array}{r} r_1\{ \\ r_1+r_2\{ \\ \vdots \\ \sum\limits_{i=1}^{j} r_i\{ \\ \rho\{ \\ \vdots \\ \rho\{ \end{array} \left\| \begin{array}{ccccccc} \overbrace{0}^{r_1} & \overbrace{0}^{r_1+r_2} & \ldots & \overbrace{0}^{\sum_{i=1}^{j} r_i} & \overbrace{0}^{\rho} & \ldots & \overbrace{-Q_j^0 (Q_j^\nu)^{-1}}^{\rho} \\ I_1 & 0 & \ldots & 0 & 0 & \ldots & -Q_j^1 (Q_j^\nu)^{-1} \\ \vdots & & & & & & \\ 0 & 0 & \ldots & I_j & 0 & \ldots & -Q_j^{j-1} (Q_j^\nu)^{-1} \\ 0 & 0 & \ldots & 0 & E & \ldots & -Q_j^j (Q_j^\nu)^{-1} \\ \vdots & & & & & & \\ 0 & 0 & \ldots & 0 & 0 & \ldots & E - Q_j^{\nu-1} (Q_j^\nu)^{-1} \end{array} \right\|,$$

$$B = \begin{array}{r} r_1\{ \\ r_1+r_2\{ \\ \vdots \\ \sum\limits_{i=1}^{j} r_i\{ \\ \rho\{ \\ \vdots \\ \rho\{ \end{array} \left\| \begin{array}{c} \overbrace{N_j^0}^{m} \\ N_j^1 \\ \vdots \\ N_j^{j-1} \\ N_j^j \\ \vdots \\ N_j^{\nu-1} \end{array} \right\|, \qquad M = \begin{array}{r} r_1\{ \\ r_1+r_2\{ \\ \vdots \\ \sum\limits_{i=1}^{j} r_i\{ \\ \rho\{ \\ \vdots \\ \rho\{ \end{array} \left\| \begin{array}{c} \overbrace{L_j^0}^{\mu} \\ L_j^1 \\ \vdots \\ L_j^{j-1} \\ L_j^j \\ \vdots \\ L_j^{\nu-1} \end{array} \right\|,$$

$$D = \bar{D}\Omega, \quad \Omega = \rho\{\| \overbrace{0}^{r_1}, \overbrace{0}^{r_1+r_2}, \ldots, \overbrace{0}^{\sum_{i=1}^{j} r_i}, \overbrace{0}^{\rho}, \ldots, \overbrace{(Q_j^\nu)^{-1}}^{\rho} \|,$$

$I_1, \ldots, I_k$ are blocks made up of the first $\sum\limits_{i=1}^{k} r_i$ columns and $\sum\limits_{i=1}^{k+1} r_i$ rows of a unit matrix $k = 1, \ldots, j$.

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